

Separability in cohomogeneity-2 Kerr-NUT-AdS metrics

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ABSTRACT: The remarkable and unexpected separability of the Hamilton-Jacobi and Klein-Gordon equations in the background of a rotating four-dimensional black hole played an important rôle in the construction of generalisations of the Kerr metric, and in the uncovering of hidden symmetries associated with the existence of Killing tensors. In this paper, we show that the Hamilton-Jacobi and Klein-Gordon equations are separable in Kerr-AdS backgrounds in all dimensions, if one specialises the rotation parameters so that the metrics have cohomogeneity 2. Furthermore, we show that this property of separability extends to the NUT generalisations of these cohomogeneity-2 black holes that we obtained in a recent paper. In all these cases, we also construct the associated irreducible rank-2 Killing tensor whose existence reflects the hidden symmetry that leads to the separability. We also consider some cohomogeneity-1 specialisations of the new Kerr-NUT-AdS metrics, showing how they relate to previous results in the literature.

KEYWORDS: Black Holes in String Theory, Integrable Equations in Physics, Classical Theories of Gravity.

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1. Introduction

In order to obtain explicit solutions to the Einstein equations, or coupled Einstein/matter equations, it is generally necessary to make simplifying symmetry assumptions about the form the metric. In some cases, where a high degree of symmetry is assumed, this alone can be sufficient to render the reduced system of equations solvable. A typical example is when one considers an ansatz for cohomogeneity-1 metrics, meaning that the remaining metric functions depend non-trivially on only a single coordinate, and hence the Einstein equations reduce to a system of ordinary differential equations.

In more complicated circumstances, it may be that symmetries of a less manifest nature can play an important rôle in allowing one to construct an explicit solution to the Einstein equations. A nice example of this kind is provided by the Kerr solution for a four-dimensional rotating black hole [1]. This is a metric of cohomogeneity 2, with non-trivial coordinate dependence on both a radial and an angular variable. It was observed, after the original discovery of the solution, that it exhibits the remarkable property, associated with a “hidden symmetry,” of allowing the separability of the Hamilton-Jacobi equation and the Klein-Gordon equation. In fact, it can be shown that the separability is related to the existence of a 2-index Killing tensor $K_{\mu\nu}$ in the Kerr geometry, satisfying $\nabla_{(\mu} K_{\nu\rho)} = 0$. By exploiting this property, and conjecturing that it would continue to hold for the more general situation with a cosmological constant, Carter was able to construct the solution

for a four-dimensional asymptotically AdS rotating black hole [2]. Further generalisations, with the inclusion, for example, of a NUT parameter, were subsequently found, again preserving the property of separability.

It is of considerable interest to investigate the issue of separability in other gravitational solutions, including in particular solutions describing black holes in higher dimensions. Not only can this shed light on the existence of hidden symmetries, associated with the existence of Killing tensors, in the known black hole metrics; it can also point the way to constructing more general solutions with additional parameters.

The general D -dimensional asymptotically-flat uncharged rotating black hole was constructed in [3], and the generalisation to the asymptotically-AdS case with a cosmological constant was constructed in [4, 5], extending an earlier result in [6] that was specific to five dimensions. In both cases, there are $[(D - 1)/2]$ independent rotation parameters a_i , characterising angular momenta in orthogonal spatial 2-planes. The general metrics are of cohomogeneity $[D/2]$, with principal orbits $\mathbb{R} \times U(1)^{[(D-1)/2]}$. It was shown in [7] that the Hamilton-Jacobi and Klein-Gordon equations are separable in all odd dimensions D , if one make the specialisation that all $(D - 1)/2$ rotation parameters a_i are set equal. This has the effect of enhancing the symmetry of the principal orbits from $\mathbb{R} \times U(1)^{(D - 1)/2}$ to $\mathbb{R} \times U((D-1)/2)$, and reducing the cohomogeneity from $(D-1)/2$ to 1. In fact, the enhanced manifest symmetry in this case is already sufficient to permit the separability, without the need for any additional hidden symmetry. Indeed, it was shown in [7] that the Killing tensor in this case is reducible, being a linear combination of direct products of Killing vectors. A non-trivial, irreducible, Killing tensor was found to exist in the case where all rotation parameters except one are vanishing [9]. Irreducible Killing tensors were also shown to exist in the special case of the five-dimensional asymptotically flat metric of [3], for arbitrary values of the two rotation parameters [8]. In the case of rotating AdS black holes in five dimensions, the separability, and associated irreducible Killing tensor, were found in [10].

A feature common to all the known cases exhibiting the phenomenon of separability is that the metric in question is of cohomogeneity ≤ 2 . A natural next step, in the investigation of separability, is therefore to examine all the D -dimensional rotating black holes under the appropriate specialisation of parameters that reduces their cohomogeneity from $[D/2]$ to 2. In fact, the D -dimensional rotating AdS black holes with this specialisation were studied recently in [11], and it was shown that they admit a generalisation in which a NUT parameter is introduced. Specifically, the specialisation that reduces the cohomogeneity to 2 is achieved by taking sets of rotation parameters to be equal in an appropriate way.

In odd dimensions, $D = 2n + 1$, cohomogeneity 2 is achieved by dividing the $n = p + q$ rotation parameters a_i into two sets, with p of them equal to a , and the remaining q parameters equal to b . At the same time, the isometry group enlarges from $\mathbb{R} \times U(1)^{p+q}$ to $\mathbb{R} \times U(p) \times U(q)$.

In even dimensions $D = 2n$, cohomogeneity 2 is achieved by instead dividing the $n - 1 = p + q$ rotation parameters into a set of p that are taken to equal a , with the remaining q parameters taken to be zero. In this case, the isometry group enlarges from $\mathbb{R} \times U(1)^{p+q}$ to $\mathbb{R} \times U(p) \times SO(2q + 1)$.

In this paper, we shall show that all these cohomogeneity-2 Kerr-AdS metrics have the property that the Hamilton-Jacobi equation and the Klein-Gordon equation are separable. Furthermore, we show that this property persists when the NUT parameter introduced in [11] is included. We also obtain the 2-index Killing tensor $K_{\mu\nu}$ that is associated with the hidden symmetry responsible for allowing the equations to separate. Unlike the case of the further specialisation to cohomogeneity 1 in odd dimensions that was studied in [7], in these cohomogeneity-2 cases the Killing tensor is irreducible.

We also study some further properties of the cohomogeneity-2 Kerr-AdS-NUT metrics that were obtained in [11]. In particular, we examine the case where one adjusts the NUT parameter so that the two adjacent roots of the metric function whose vanishing defines the endpoints of the range of one of the inhomogeneous coordinates become coincident. After appropriate scalings, this limit yields NUT-type metrics of cohomogeneity 1, which in some special cases coincide with NUT generalisations obtained previously in [12].

2. Separability in $D = 2n$ dimensions

It was shown in [11] that if one sets p rotation parameters a_i equal to a , and the remaining q rotation parameters to zero in general Kerr-AdS metrics of [4, 5] in dimension $D = 2n$, where $p + q = n - 1$, then one can introduce a NUT parameter L to complement the mass parameter M , with the metric being given by

$$ds^2 = \frac{r^2 + v^2}{X} dr^2 + \frac{r^2 + v^2}{Y} dv^2 - \frac{X}{r^2 + v^2} \left(dt - \frac{a^2 - v^2}{a \Xi_a} (d\psi + A) \right)^2 \tag{2.1}$$

$$+ \frac{Y}{r^2 + v^2} \left(dt - \frac{a^2 + r^2}{a \Xi_a} (d\psi + A) \right)^2 + \frac{(a^2 + r^2)(a^2 - v^2)}{a^2 \Xi_a} d\Sigma_{p-1}^2 + \frac{r^2 v^2}{a^2} d\Omega_{2q}^2.$$

Here, $d\Omega_{2q}^2$ is the metric on the unit sphere S^{2q} , $d\Sigma_{p-1}^2$ is the standard Fubini-Study metric on the “unit” complex projective space $\mathbb{C}\mathbb{P}^{p-1}$ with Kähler form $J = \frac{1}{2}dA$, and the metric functions X and Y are given by

$$X = (1 + g^2 r^2)(r^2 + a^2) - \frac{2Mr}{(r^2 + a^2)^{p-1} r^{2q}},$$

$$Y = (1 - g^2 v^2)(a^2 - v^2) - \frac{2Lv}{(a^2 - v^2)^{p-1} v^{2q}}. \tag{2.2}$$

It should be noted that one can replace the unit-sphere metric $d\Omega_{2q}^2 = \gamma_{ij} dx^i dx^j$ by any $(2q)$ -dimensional Einstein metric normalised to $R_{ij} = (2q - 1) \gamma_{ij}$, and the Fubini-Study metric $d\Sigma_{p-1}^2 = h_{mn} dx^m dx^n$ on $\mathbb{C}\mathbb{P}^{p-1}$ can be replaced by any Einstein-Kähler $(2p - 2)$ -metric normalised to $R_{mn} = 2p h_{mn}$, and one again has a local solution of the Einstein equations.

It is not hard to see that the inverse of the metric (2.1), which we can write as $(\partial/\partial s)^2 \equiv g^{\mu\nu} \partial_\mu \partial_\nu$, is given by

$$(r^2 + v^2) \left(\frac{\partial}{\partial s} \right)^2 = X \left(\frac{\partial}{\partial r} \right)^2 + Y \left(\frac{\partial}{\partial v} \right)^2 - \frac{1}{X} \left((r^2 + a^2) \frac{\partial}{\partial t} + a \Xi_a \frac{\partial}{\partial \psi} \right)^2$$

$$\begin{aligned}
& + \frac{1}{Y} \left((a^2 - v^2) \frac{\partial}{\partial t} + a \Xi_a \frac{\partial}{\partial \psi} \right)^2 + a^2 \left(\frac{1}{r^2} + \frac{1}{v^2} \right) \left(\frac{\partial}{\partial \Omega} \right)^2 \\
& - a^2 \Xi_a \left(\frac{1}{r^2 + a^2} - \frac{1}{a^2 - v^2} \right) h^{mn} \left(\partial_m - A_m \frac{\partial}{\partial \psi} \right) \left(\partial_n - A_n \frac{\partial}{\partial \psi} \right),
\end{aligned} \tag{2.3}$$

where A_m are the components of the 1-form A , $(\frac{\partial}{\partial \Omega})^2 = \gamma^{ij} \partial_i \partial_j$ is the inverse of the metric on the unit $2q$ -sphere, and h^{mn} are the components of the inverse of the Fubini-Study metric on $\mathbb{C}\mathbb{P}^{p-1}$.

2.1 Separability of the Hamilton-Jacobi equation

The covariant Hamiltonian function on the cotangent bundle of the metric (2.1) is given by

$$\mathcal{H}(\mathcal{P}_\mu, x^\mu) \equiv \frac{1}{2} g^{\mu\nu} \mathcal{P}_\mu \mathcal{P}_\nu, \tag{2.4}$$

where \mathcal{P}_μ are the canonical momenta conjugate to the coordinates x^μ . In terms of Hamilton's principle function S , one has $\mathcal{P}_\mu = \partial_\mu S$, and the Hamilton-Jacobi equation is given by

$$\mathcal{H}(\partial_\mu S, x^\mu) = -\frac{1}{2} \mu^2. \tag{2.5}$$

It is evident from (2.3) that the Hamilton-Jacobi equation admits separable solutions of the form

$$S = -Et + J_\psi \psi + F(r) + G(v) + P + Q, \tag{2.6}$$

where P is a function of the $\mathbb{C}\mathbb{P}^{p-1}$ coordinates only, and Q is a function of the S^{2p} coordinates only. We introduce separation constants K_Σ and K_Ω for the functions on these spaces, so that

$$\left(\frac{\partial Q}{\partial \Omega} \right)^2 = K_\Omega^2, \quad h^{mn} (\partial_m P - J_\psi A_m P) (\partial_n P - J_\psi A_n P) = K_\Sigma^2. \tag{2.7}$$

From the above, we can read off the remaining non-trivial equations for the functions $F(r)$ and $G(v)$ in (2.6), finding

$$\begin{aligned}
-2\kappa &= XF'^2 - \frac{1}{X} \left(E(r^2 + a^2) - a \Xi_a J_\psi \right)^2 + \frac{a^2 K_\Omega^2}{r^2} - \frac{a^2 \Xi_a K_\Sigma^2}{r^2 + a^2} + \mu^2 r^2, \\
2\kappa &= Y\dot{G}^2 + \frac{1}{Y} \left(E(a^2 - v^2) - a \Xi_a J_\psi \right)^2 + \frac{a^2 K_\Omega^2}{v^2} + \frac{a^2 \Xi_a K_\Sigma^2}{a^2 - v^2} + \mu^2 v^2,
\end{aligned} \tag{2.8}$$

where F' denotes dF/dr , \dot{G} denotes dG/dv , and κ is the separation constant associated with the non-trivial hidden symmetry that permits the separation of the Hamilton-Jacobi equation.

Note that the separation demonstrated thus far works equally well if $d\Sigma_{p-1}^2$ is any $(2p-2)$ -dimensional Einstein-Kähler metric and $d\Omega_{2q}^2$ is any Einstein metric with the same scalar curvatures as the $\mathbb{C}\mathbb{P}^{p-1}$ and S^{2q} metrics respectively. A complete separability, in which the functions P and Q are themselves fully separated, depends upon the complete separability of the Hamilton-Jacobi equations in these two spaces. In particular, this is possible whenever they are homogeneous spaces, as is the case for $\mathbb{C}\mathbb{P}^{p-1}$ and S^{2q} . Note

that in the case of the Einstein-Kähler space, the relevant Hamilton-Jacobi equation is the one describing a particle of charge J_ψ in geodesic motion, with minimal coupling to the potential A whose field strength is $2J$, where J is the Kähler form.

Following the discussion in [9], we note that associated with the separation constant κ is a Poisson function \mathcal{K} , which Poisson commutes with the Hamiltonian \mathcal{H} . The function \mathcal{K} is equal to the separation constant κ if the Hamilton-Jacobi equations are satisfied, and so we can simply read it off from either of the equations in (2.8), or any linear combination thereof. Thus, for example, from the first equation in (2.8) we may read off

$$\begin{aligned} \mathcal{K} = & -\frac{1}{2}X\mathcal{P}_r^2 + \frac{1}{2X}\left((r^2 + a^2)\mathcal{P}_t + a\Xi_a\mathcal{P}_\psi\right)^2 - \frac{a^2\mathcal{P}_\Omega^2}{2r^2} + \frac{1}{2}r^2g^{\mu\nu}\mathcal{P}_\mu\mathcal{P}_\nu \\ & + \frac{a^2\Xi_a}{2(r^2 + a^2)}h^{mn}(\mathcal{P}_m - A_m\mathcal{P}_\psi)(\mathcal{P}_n - A_n\mathcal{P}_\psi), \end{aligned} \quad (2.9)$$

where $\mathcal{P}_\Omega^2 \equiv \gamma^{ij}\mathcal{P}_i\mathcal{P}_j$ and γ^{ij} is the inverse metric on the unit sphere S^{2q} . An alternative way of writing \mathcal{K} , which puts the r and v coordinates on an equivalent footing, is to take the linear combination of the two equations in (2.8) that eliminates μ^2 , yielding

$$\begin{aligned} \mathcal{K} = & \frac{1}{2(r^2 + v^2)}\left[\frac{v^2}{X}\left[(r^2 + a^2)\mathcal{P}_t + a\Xi_a\mathcal{P}_\psi\right]^2 + \frac{r^2}{Y}\left[(a^2 - v^2)\mathcal{P}_t + a\Xi_a\mathcal{P}_\psi\right]^2\right. \\ & \left. - v^2X\mathcal{P}_r^2 + r^2Y\mathcal{P}_v^2\right] + \frac{a^2}{2}\left(\frac{1}{v^2} - \frac{1}{r^2}\right)\mathcal{P}_\Omega^2 \\ & + \frac{a^2 + r^2 - v^2}{2(r^2 + a^2)(a^2 - v^2)}h^{mn}(\mathcal{P}_m - A_m\mathcal{P}_\psi)(\mathcal{P}_n - A_n\mathcal{P}_\psi), \end{aligned} \quad (2.10)$$

The function \mathcal{K} defines a Stäckel-Killing tensor with components $K^{\mu\nu}$, given by

$$\mathcal{K} = \frac{1}{2}K^{\mu\nu}\mathcal{P}_\mu\mathcal{P}_\nu \quad (2.11)$$

Thus the components $K^{\mu\nu}$ can be read off trivially from (2.9) or (2.10) by inspection. The Stäckel-Killing tensor satisfies

$$\nabla_{(\mu}K_{\nu\rho)} = 0, \quad (2.12)$$

by virtue of the fact that \mathcal{K} Poisson commutes with \mathcal{H} .

2.2 Separability of the Klein-Gordon equation

The separability of the Klein-Gordon equation is closely related to that of the Hamilton-Jacobi equation. A key observation, which can easily be seen from (2.1), is that

$$\sqrt{-g} = a^{1-2p-2q}\Xi_a^{-p}(r^2 + a^2)^{p-1}(a^2 - v^2)^{p-1}r^{2q}v^{2q}\sqrt{h}\sqrt{\gamma}(r^2 + v^2). \quad (2.13)$$

Aside from the factor $(r^2 + v^2)$, the coordinate dependence of $\sqrt{-g}$ therefore factorises into a product of a function of r , a function of v , a function of the S^{2q} coordinates and a function of the $\mathbb{C}\mathbb{P}^{p-1}$ coordinates. Since the Laplacian is given by

$$\square = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right), \quad (2.14)$$

it follows from (2.3) that the Klein-Gordon equation $\square f = \lambda f$ becomes

$$\begin{aligned}
 & \frac{1}{(r^2 + a^2)^{p-1} r^{2q}} \frac{\partial}{\partial r} \left((r^2 + a^2)^{p-1} r^{2q} X \frac{\partial f}{\partial r} \right) \\
 & + \frac{1}{(a^2 - v^2)^{p-1} v^{2q}} \frac{\partial}{\partial v} \left((a^2 - v^2)^{p-1} v^{2q} Y \frac{\partial f}{\partial v} \right) \\
 & - \frac{1}{X} \left((r^2 + a^2) \frac{\partial}{\partial t} + a \Xi_a \frac{\partial}{\partial \psi} \right)^2 f + \frac{1}{Y} \left((a^2 - v^2) \frac{\partial}{\partial t} + a \Xi_a \frac{\partial}{\partial \psi} \right)^2 f \quad (2.15) \\
 & + a^2 \left(\frac{1}{r^2} + \frac{1}{v^2} \right) \frac{1}{\sqrt{\gamma}} \partial_i (\sqrt{\gamma} \gamma^{ij} \partial_j f) \\
 & - a^2 \Xi_a \left(\frac{1}{r^2 + a^2} - \frac{1}{a^2 - v^2} \right) \frac{1}{\sqrt{h}} D_m (\sqrt{h} h^{mn} D_n f) = \lambda (r^2 + v^2) f,
 \end{aligned}$$

where $D_m \equiv \partial_m - A_m \partial/\partial \psi$. It is manifest that the equation can be separated by writing f as a product of functions of r, v, ψ , the S^{2q} coordinates and the \mathbb{CP}^{p-1} coordinates. Of course the complete separability of the equation depends upon the fact that one can fully separate the Klein-Gordon equations on S^{2q} and \mathbb{CP}^{p-1} , by virtue of the homogeneity of these spaces.

3. Separability in $D = 2n + 1$ dimensions

It was shown in [11] that if one sets p rotation parameters a_i equal to a , and the remaining q rotation parameters equal to b in general Kerr-AdS metrics of [4, 5] in dimension $D = 2n + 1$, where $p + q = n$, then one can introduce a NUT parameter L to complement the mass parameter M , with the metric being given by

$$\begin{aligned}
 ds^2 = & \frac{r^2 + v^2}{X} dr^2 + \frac{r^2 + v^2}{Y} dv^2 + \frac{(r^2 + a^2)(a^2 - v^2)}{\Xi_a(a^2 - b^2)} d\Sigma_{p-1}^2 + \frac{(r^2 + b^2)(b^2 - v^2)}{\Xi_b(b^2 - a^2)} d\tilde{\Sigma}_{q-1}^2 \\
 & + \frac{a^2 b^2}{r^2 v^2} \left[dt - (r^2 - v^2) d\phi - r^2 v^2 d\varphi - \frac{(r^2 + a^2)(a^2 - v^2)}{a \Xi_a(a^2 - b^2)} A - \frac{(r^2 + b^2)(b^2 - v^2)}{b \Xi_b(b^2 - a^2)} B \right]^2 \\
 & - \frac{X}{r^2 + v^2} \left[dt + v^2 d\phi - \frac{a(a^2 - v^2)}{\Xi_a(a^2 - b^2)} A - \frac{b(b^2 - v^2)}{\Xi_b(b^2 - a^2)} B \right]^2 \\
 & + \frac{Y}{r^2 + v^2} \left[dt - r^2 d\phi - \frac{a(r^2 + a^2)}{\Xi_a(a^2 - b^2)} A - \frac{b(r^2 + b^2)}{\Xi_b(b^2 - a^2)} B \right]^2, \quad (3.1)
 \end{aligned}$$

where

$$\begin{aligned}
 X = & \frac{(1 + g^2 r^2)(r^2 + a^2)(r^2 + b^2)}{r^2} - \frac{2M}{(r^2 + a^2)^{p-1} (r^2 + b^2)^{q-1}}, \\
 Y = & \frac{-(1 - g^2 v^2)(a^2 - v^2)(b^2 - v^2)}{v^2} + \frac{2L}{(a^2 - v^2)^{p-1} (b^2 - v^2)^{q-1}}. \quad (3.2)
 \end{aligned}$$

Here, $d\Sigma_{p-1}^2$ and $d\tilde{\Sigma}_{q-1}^2$ are the standard ‘‘unit’’ metrics on two complex projective spaces \mathbb{CP}^{p-1} and \mathbb{CP}^{q-1} , with Kähler forms given locally by $J = \frac{1}{2} dA$ and $\tilde{J} = \frac{1}{2} dB$. One can also obtain more general solutions by replacing the complex projective spaces with their Fubini-Study metrics by any other Einstein-Kähler metrics with the same Ricci scalars.

One can straightforwardly show that the inverse $(\partial/\partial s)^2$ of the metric (3.1) is given by

$$\begin{aligned}
(r^2 + v^2) \left(\frac{\partial}{\partial s} \right)^2 &= X \left(\frac{\partial}{\partial r} \right)^2 + Y \left(\frac{\partial}{\partial v} \right)^2 + \frac{1}{a^2 b^2} \left(\frac{1}{r^2} + \frac{1}{v^2} \right) \left(\frac{\partial}{\partial \varphi} \right)^2 \\
&\quad - \frac{1}{X} \left(r^2 \frac{\partial}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \phi} \right)^2 + \frac{1}{Y} \left(v^2 \frac{\partial}{\partial t} + \frac{1}{v^2} \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \phi} \right)^2 \\
&\quad - (a^2 - b^2) \Xi_a \left(\frac{1}{r^2 + a^2} - \frac{1}{a^2 - v^2} \right) h^{mn} D_m D_n \\
&\quad - (b^2 - a^2) \Xi_b \left(\frac{1}{r^2 + b^2} - \frac{1}{b^2 - v^2} \right) \tilde{h}^{k\ell} \tilde{D}_k \tilde{D}_\ell,
\end{aligned} \tag{3.3}$$

where

$$\begin{aligned}
D_m &\equiv \partial_m - \frac{a A_m}{(a^2 - b^2) \Xi_a} \left(\frac{\partial}{\partial \phi} - a^2 \frac{\partial}{\partial t} - \frac{1}{a^2} \frac{\partial}{\partial \varphi} \right), \\
\tilde{D}_k &\equiv \partial_k - \frac{b B_k}{(b^2 - a^2) \Xi_b} \left(\frac{\partial}{\partial \phi} - b^2 \frac{\partial}{\partial t} - \frac{1}{b^2} \frac{\partial}{\partial \varphi} \right),
\end{aligned} \tag{3.4}$$

and h^{mn} and $\tilde{h}^{k\ell}$ are the inverses of the Fubini-Study metrics $d\Sigma_{p-1}^2$ and $d\Sigma_{q-1}^2$ on the complex projective spaces $\mathbb{C}\mathbb{P}^{p-1}$ and $\mathbb{C}\mathbb{P}^{q-1}$.

3.1 Separability of the Hamilton-Jacobi equation

Following analogous steps to those we described in section 2, it can be seen that the Hamilton-Jacobi equation is separable, if we write the Hamilton principle function as

$$S = -Et + J_\varphi \varphi + J_\phi \phi + F(r) + G(v) + P + \tilde{P}, \tag{3.5}$$

where P depends only on the coordinates of $\mathbb{C}\mathbb{P}^{p-1}$, and \tilde{P} depends only on the coordinates of $\mathbb{C}\mathbb{P}^{q-1}$. We have separation constants K_Σ and $K_{\tilde{\Sigma}}$ associated with the two complex projective space factors. The Hamilton-Jacobi equations in these two subspaces themselves describe particles of charges q and \tilde{q} minimally coupled to the vector potentials A and B respectively, where

$$q = \frac{a}{(a^2 - b^2) \Xi_a} \left(J_\phi + a^2 E - \frac{1}{a^2} J_\varphi \right), \quad \tilde{q} = \frac{b}{(b^2 - a^2) \Xi_b} \left(J_\phi + b^2 E - \frac{1}{b^2} J_\varphi \right), \tag{3.6}$$

and

$$h^{mn} (\partial_m P - q A_m) (\partial_n P - q A_n) = K_\Sigma^2, \quad \tilde{h}^{k\ell} (\partial_k \tilde{P} - \tilde{q} A_k) (\partial_\ell \tilde{P} - \tilde{q} A_\ell) = K_{\tilde{\Sigma}}^2. \tag{3.7}$$

From (3.3), it then follows that there is a further non-trivial separation constant κ , leading to the equations

$$\begin{aligned}
-2\kappa &= X F'^2 + \frac{J_\varphi^2}{a^2 b^2 r^2} - \frac{1}{X} \left(E r^2 - \frac{1}{r^2} J_\varphi - J_\phi \right)^2 \\
&\quad - \frac{(a^2 - b^2) \Xi_a K_\Sigma^2}{r^2 + a^2} - \frac{(b^2 - a^2) \Xi_b K_{\tilde{\Sigma}}^2}{r^2 + b^2} + \mu^2 r^2, \\
2\kappa &= Y \dot{G}^2 + \frac{J_\varphi^2}{a^2 b^2 v^2} + \frac{1}{Y} \left(E v^2 - \frac{1}{v^2} J_\varphi + J_\phi \right)^2 \\
&\quad + \frac{(a^2 - b^2) \Xi_a K_\Sigma^2}{a^2 - v^2} + \frac{(b^2 - a^2) \Xi_b K_{\tilde{\Sigma}}^2}{b^2 - v^2} + \mu^2 v^2.
\end{aligned} \tag{3.8}$$

We can then read off the associated Poisson function \mathcal{K} that commutes with the Hamiltonian \mathcal{H} , and which takes the constant value κ upon use of the Hamilton-Jacobi equations. As in section 2, one can organise the expression for \mathcal{K} in different ways, depending on the choice of linear combination of the two expressions in (3.8) that one makes. Thus, for example, from the first expression we can write \mathcal{K} as

$$\begin{aligned} \mathcal{K} = & -\frac{1}{2}X\mathcal{P}_r^2 - \frac{1}{2a^2b^2r^2}\mathcal{P}_\varphi^2 + \frac{1}{2X}\left(r^2\mathcal{P}_t + \frac{1}{r^2}\mathcal{P}_\varphi + \mathcal{P}_\phi\right)^2 \\ & + \frac{(a^2 - b^2)\Xi_a}{2(r^2 + a^2)}\mathcal{P}_\Sigma^2 + \frac{(b^2 - a^2)\Xi_b}{2(r^2 + b^2)}\mathcal{P}_\Sigma^2 + \frac{1}{2}r^2 g^{\mu\nu} \mathcal{P}_\mu \mathcal{P}_\nu, \end{aligned} \quad (3.9)$$

where

$$\begin{aligned} \mathcal{P}_\Sigma^2 \equiv & h^{mn} \left[\mathcal{P}_m - \frac{aA_m}{(a^2 - b^2)\Xi_a} \left(\mathcal{P}_\phi - a^2\mathcal{P}_t - \frac{1}{a^2}\mathcal{P}_\varphi \right) \right] \\ & \times \left[\mathcal{P}_n - \frac{aA_n}{(a^2 - b^2)\Xi_a} \left(\mathcal{P}_\phi - a^2\mathcal{P}_t - \frac{1}{a^2}\mathcal{P}_\varphi \right) \right], \\ \mathcal{P}_\Sigma^2 \equiv & \tilde{h}^{k\ell} \left[\mathcal{P}_k - \frac{bB_k}{(b^2 - a^2)\Xi_b} \left(\mathcal{P}_\phi - b^2\mathcal{P}_t - \frac{1}{b^2}\mathcal{P}_\varphi \right) \right] \\ & \times \left[\mathcal{P}_\ell - \frac{bB_\ell}{(b^2 - a^2)\Xi_b} \left(\mathcal{P}_\phi - b^2\mathcal{P}_t - \frac{1}{b^2}\mathcal{P}_\varphi \right) \right]. \end{aligned} \quad (3.10)$$

The components of the associated Killing tensor $K_{\mu\nu}$ can be read off directly from (3.9), via $\mathcal{K} = \frac{1}{2}K^{\mu\nu}\mathcal{P}_\mu\mathcal{P}_\nu$. Again, as in the even-dimensional case discussed in section 2, one can equivalently express \mathcal{K} in a more symmetrical fashion by taking the linear combination of the two equations in (3.8) that eliminates μ^2 .

3.2 Separability of the Klein-Gordon equation

As in the case of even dimensions, here too the separability of the Klein-Gordon equation is closely related to the separability of the Hamilton-Jacobi equation. Again, the key point is that $\sqrt{-g}$ has a simple form, being proportional to $(r^2 + v^2)$ times a product of functions of r , v and the coordinates on the two complex projective spaces:

$$\sqrt{-g} = \frac{abr v \sqrt{\tilde{h}} \sqrt{\tilde{h}}}{|a^2 - b^2|^{n-2} \Xi_a^{p-1} \Xi_b^{q-1}} (r^2 + a^2)^{p-1} (r^2 + b^2)^{q-1} (a^2 - v^2)^{p-1} (b^2 - v^2)^{q-1} (r^2 + v^2). \quad (3.11)$$

Together with the the expression (3.3) for the inverse metric, we see that the Klein-Gordon equation $\square f = \lambda f$ assumes the manifestly separable form

$$\begin{aligned} & \frac{1}{r(r^2 + a^2)^{p-1}(r^2 + b^2)^{q-1}} \frac{\partial}{\partial r} \left(r(r^2 + a^2)^{p-1}(r^2 + b^2)^{q-1} X \frac{\partial f}{\partial r} \right) \\ & \frac{1}{v(a^2 - v^2)^{p-1}(b^2 - v^2)^{q-1}} \frac{\partial}{\partial v} \left(v(a^2 - v^2)^{p-1}(b^2 - v^2)^{q-1} Y \frac{\partial f}{\partial v} \right) \\ & - \frac{1}{X} \left(r^2 \frac{\partial}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial \varphi} + \frac{\partial}{\partial \phi} \right)^2 f + \frac{1}{Y} \left(v^2 \frac{\partial}{\partial t} + \frac{1}{v^2} \frac{\partial}{\partial \varphi} - \frac{\partial}{\partial \phi} \right)^2 f + \frac{1}{a^2 b^2} \left(\frac{1}{r^2} + \frac{1}{v^2} \right) \frac{\partial^2 f}{\partial \varphi^2} \end{aligned}$$

$$\begin{aligned}
 & -(a^2 - b^2)\Xi_a \left(\frac{1}{r^2 + a^2} - \frac{1}{a^2 - v^2} \right) \frac{1}{\sqrt{h}} D_m (\sqrt{h} h^{mn} D_n f) \\
 & -(b^2 - a^2)\Xi_b \left(\frac{1}{r^2 + b^2} - \frac{1}{b^2 - v^2} \right) \frac{1}{\sqrt{\tilde{h}}} \tilde{D}_k (\sqrt{\tilde{h}} \tilde{h}^{kl} \tilde{D}_\ell f) = \lambda(r^2 + v^2)f.
 \end{aligned} \tag{3.12}$$

Note that D_m and \tilde{D}_k , defined in (3.4), yield gauge covariant derivatives acting on charged wavefunctions in the two complex projective spaces, once one separates variables by writing f as a product of functions of the coordinates. As in the previous discussions, the complete separability of the system depends upon the separability of the Klein-Gordon equations in the complex projective spaces.

4. Specialisation to NUT metrics of cohomogeneity 1

The new NUT generalisations of the Kerr-AdS metrics that were found in [11] all have cohomogeneity 2, and, as we have shown in this paper, they all share the feature that the Hamilton-Jacobi equation and the Klein-Gordon equation are separable in these backgrounds. It is also of interest to see how these cohomogeneity-2 Kerr-NUT-AdS metrics reduce to certain previously-known solutions under specialisations of the parameters. In particular, we shall show that if one applies a limiting procedure in which the cohomogeneity is reduced from 2 to 1, then the resulting metrics include some higher-dimensional NUT metrics that were obtained in [12]. As usual, the discussion divides into the cases of even-dimensional metrics and odd-dimensional metrics.

4.1 $D = 2n$

Our starting point is the class of new even-dimensional Kerr-NUT-AdS metrics that were obtained in [11], and which take the form given in (2.1). The cohomogeneity can be reduced from 2 to 1 by specialising the parameters in such a way that the two adjacent roots of the function $Y(v)$ that define the range of the v coordinate become coincident. Provided the v coordinate is rescaled appropriately as the limit is taken, one obtains a non-singular metric that now no longer has any dependence on the rescaled v coordinate.

The function $Y(v)$ acquires a double root, at $v = v_0$, if the parameters a and L are chosen to satisfy

$$\begin{aligned}
 L = L_0 & \equiv \frac{(a^2 - v_0^2)^{p+1} v_0^{2q-1}}{(2q+1)a^2 - (2p+2q+1)v_0^2}, \\
 g^2 & = \frac{(2q-1)a^2 - (2p+2q-1)v_0^2}{v_0^2((2q+1)a^2 - (2p+2q+1)v_0^2)}.
 \end{aligned} \tag{4.1}$$

In order to approach this limit with an appropriately rescaled v coordinate, we define

$$v = v_0 + \epsilon \cos \chi, \quad L = L_0(1 + \epsilon^2 c), \tag{4.2}$$

with the constant c given by

$$c = \frac{a^4(1 - 4q^2) + 2a^2(2q - 1)(2p + 2q + 1)v_0^2 + (1 - 4(p + q)^2)v_0^4}{2(a^2 - v_0^2)^2 v_0^2}, \tag{4.3}$$

where ϵ will shortly be sent to zero. The function Y under this limit becomes

$$Y = \epsilon^2 Y_0 \sin^2 \chi, \tag{4.4}$$

where

$$Y_0 = \frac{2(a - v_0)^2 c}{(2p + 2q + 1)v_0^2 - (2q + 1)a^2}. \tag{4.5}$$

In order for the metric (2.1) to be nonsingular in the limit, we must also make the coordinate transformations

$$\psi \rightarrow \frac{a \Xi_a}{\epsilon Y_0} \tilde{\psi}, \quad t \rightarrow t + \frac{a^2 - v_0^2}{\epsilon Y_0} \psi. \tag{4.6}$$

Sending ϵ to zero, the metric (2.1) then becomes

$$ds^2 = -\frac{X}{r^2 + v_0^2} \left(dt + \frac{2v_0}{Y_0} \cos \chi d\psi - \frac{a^2 - v_0^2}{a \Xi_a} A \right)^2 + \frac{r^2 + v_0^2}{X} dr^2 + \frac{(r^2 + v_0^2)}{Y_0} (d\chi^2 + \sin^2 \chi d\psi^2) + \frac{r^2 + a^2}{a^2 \Xi_a} d\Sigma_{p-1}^2 + \frac{r^2 v_0^2}{a^2} d\Omega_{2q}^2. \tag{4.7}$$

The metrics (4.7) are contained within a rather general class of cohomogeneity-1 NUT metrics that were obtained in [12]. The case $p = 1$ and $q = 0$ reduces to the standard Taub-NUT-AdS metric in four dimensions.

4.2 $D = 2n + 1$

In odd dimensions, our starting point is the cohomogeneity-2 Kerr-NUT-AdS metrics found in [11], and presented in equation (3.1).

Proceeding in an analogous fashion to the discussion we gave in even dimensions, we first consider the conditions under which Y has a double root, at $v = v_0$. This happens when the constants L , a and b are chosen such that

$$L = L_0 \equiv \frac{(a^2 - v_0^2)^p (b^2 - v_0^2)^q (1 - g^2 v_0^2)}{2v_0^2}, \tag{4.8}$$

$$g^2 = \frac{a^2 b^2 + (a^2(q - 1) + b^2(p - 1))v_0^2 - (p + q - 1)v_0^4}{(a^2 q + b^2 p - (p + q)v_0^2)v_0^4}.$$

Next, we deform away slightly from the double root, and introduce a new coordinate χ in place of v :

$$v = v_0 + \epsilon \cos \chi, \quad L = L_0(1 + \epsilon^2 c), \tag{4.9}$$

where the constant c is given by

$$c = \frac{2}{(a^2 - v_0^2)^2 (b^2 - v_0^2)^2} \left(-2a^2 b^2 (a^2 q + b^2 p) - (a^4 q (q - 1) + b^4 p (p - 1) + 2a^2 b^2 (pq - 2p - 2q))v_0^2 + 2(p + q)(a^2 (q - 1) + b^2 (p - 1))v_0^4 - (p + q)(p + q - 1)v_0^6 \right). \tag{4.10}$$

The function Y in this limit becomes

$$Y = \epsilon^2 Y_0 \sin^2 \chi, \tag{4.11}$$

where

$$Y_0 = \frac{(a^2 - v_0^2)^2 (b^2 - v_0^2)^2 c}{v_0^4 ((p+q)v_0^2 - a^2 q - b^2 p)}. \quad (4.12)$$

Making the further coordinate transformation

$$t \rightarrow t - \frac{v_0^2 \phi}{\epsilon Y_0}, \quad \phi \rightarrow \frac{\phi}{\epsilon Y_0}, \quad \varphi \rightarrow -\frac{\phi}{v_0^2 \epsilon Y_0} - \frac{\varphi}{v_0^4}, \quad (4.13)$$

we can now obtain a smooth limit in which ϵ is sent to zero, for which the metric (3.1) becomes

$$\begin{aligned} ds^2 = & -\frac{X}{r^2 + v_0^2} \left[dt + \frac{2v_0}{Y_0} \cos \chi d\phi - \frac{a(a^2 - v_0^2)}{\Xi_a(a^2 - b^2)} A - \frac{b(b^2 - v_0^2)}{\Xi_b(b^2 - a^2)} B \right]^2 \\ & + \frac{a^2 b^2}{r^2 v_0^2} \left[dt + \frac{2v_0}{Y_0} \cos \chi d\phi + \frac{r^2}{v_0^2} (d\varphi + \frac{2v_0}{Y_0} \cos \chi d\phi) - \frac{(r^2 + a^2)(a^2 - v_0^2)}{a \Xi_a(a^2 - b^2)} A \right. \\ & \left. - \frac{(r^2 + b^2)(b^2 - v_0^2)}{b \Xi_b(b^2 - a^2)} B \right]^2 + \frac{r^2 + v_0^2}{X} dr^2 + \frac{r^2 + v_0^2}{Y_0} (d\chi^2 + \sin^2 \chi d\phi^2) \\ & + \frac{(r^2 + a^2)(a^2 - v_0^2)}{\Xi_a(a^2 - b^2)} d\Sigma_{p-1}^2 + \frac{(r^2 + b^2)(b^2 - v_0^2)}{\Xi_b(b^2 - a^2)} d\tilde{\Sigma}_{q-1}^2. \end{aligned} \quad (4.14)$$

This metric is contained within the class of cohomogeneity-1 NUT generalisations that were considered in [12].

5. Conclusions

The separability of the Hamilton-Jacobi and Klein-Gordon equations in the background of a rotating four-dimensional black hole played an important rôle in the construction of generalisations of the Kerr metric, and in the uncovering of hidden symmetries associated with the existence of Killing tensors. In this paper, we have shown that the Hamilton-Jacobi and Klein-Gordon equations are separable in Kerr-AdS backgrounds in all dimensions, if one specialises the rotation parameters so that the metrics have cohomogeneity 2. Furthermore, we have shown that this property of separability extends to the NUT generalisations of these cohomogeneity-2 black holes that we obtained in [11]. In all these cases, we also constructed the associated irreducible rank-2 Killing tensor whose existence reflects the hidden symmetry that leads to the separability. We also considered some cohomogeneity-1 specialisations of the new Kerr-NUT-AdS metrics, and showed how they relate to previous results in the literature [12].

The results on separability that we have obtained in this paper raise the interesting question of whether it might extend to the higher-dimensional rotating black holes with more general choices for the rotation parameters, and thus having cohomogeneity larger than 2. This question is currently under investigation.

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